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Distributed dynamic consensus under quantized communication data

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SUMMARY

Distributed dynamic average consensus is investigated under quantized communication data. We use a uniform quantizer with constant quantization step-size to deal with the saturation caused by the dynamic consensus error and propose a communication feedback-based distributed consensus protocol suitable for directed time-varying topologies to make the internal state of each agent's encoder consistent with the output of its neighbors' decoder. For the case where the communication topology is directed, balanced and periodically connected, it is shown that if the difference of the reference inputs satisfies some boundedness condition, then the designed quantized dynamic consensus protocol can ensure the states of all the agents achieve dynamic average consensus with arbitrarily small steady state error by properly choosing system parameters. The lower bound of the required quantization levels and the method to choose the system parameters are also presented. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Recently, distributed consensus of multi-agent systems (MASs) has gained increasing research attention [1–11]. This topic is aimed at discussing how to design a distributed protocol based on local information to ensure that all the states of the agents converge to a common value. When the common value is the average of the initial states of all the agents, the consensus is called static average consensus; when the common value is the average of the time-varying reference inputs of all the agents, it is called dynamic average consensus, where the reference inputs may be the information such as agents' position, attitude, and temperature of the environment. Dynamic average consensus protocols have been applied to many practical areas, such as distributed tracking [12, 13], distributed estimation [14], and formation control [15].

In the substantial body of previous works, one meaningful research line is about how to design distributed consensus protocols when the communications are with constraints, such as communication noises [16, 17], packet losses [18], and energy and bandwidth limitations [19]. Sometimes, digital communication channels with limited data rate may be used by each pair of adjacent agents to exchange information. In this case, only symbol information instead of the real number sequence can be transmitted, which makes it necessary to apply some information quantization techniques.

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In the control theory field, results on distributed quantized consensus have been gradually coming up, which mainly focus on the static consensus problem [19–25]. Based on quantized data, Carli *et al.*, Frasca *et al.*, and Kashyap *et al.* [20–22, 24] developed different types of distributed consensus algorithms, such as gossip averaging protocols and average-preserving consensus protocols, to make the agents' states converge to the initial state average or its integer approximation with an error no greater than 1. However, these works require each uniform quantizer has infinite quantization levels. For the case of undirected fixed topology, Li *et al.* [19] investigated the following basic problem: to ensure an MAS to achieve consensus, how many bits of information does each pair of adjacent agents need to exchange at each time step? The authors proved that for any uniform quantizer with finite quantization levels, one can always get an average consensus with exponential convergence rate by properly choosing the system parameters. Zhang and Zhang [25] further discussed the case of time-varying topologies and proposed a communication feedback-based distributed consensus protocol to deal with the inconsistence between the internal state of each agent's encoder and the output of its neighbors' decoder. The authors also gave a finite lower bound of the communication data rate between each pair of adjacent agents to ensure the exponential consensus.

For the dynamic consensus problem, the existing works are mainly concentrating on MASs with ideal and constraint-free communication channels [12, 26–29], and the case of quantized communications has not received enough attention. For the continuous-time case, Ren [28] proposed a proportional–derivative consensus protocol and made the agents' states to track a common time-varying reference input available to only a subset of agents. Spanos *et al.* and Freeman *et al.* [12, 27] considered the case where the reference inputs of agents were different. The goal of each agent is to track the average of the time-varying signals by use of local communications with its neighbors. Precisely, by a frequency-domain method, Spanos *et al.* [12] gave a dynamic consensus protocol under undirected fixed topology and achieved a zero steady-error tracking for polynomially bounded time-varying reference inputs. Freeman *et al.* [27] proposed proportional and proportional–integral dynamic consensus protocols under time-varying topologies and showed the designed algorithms could track the average of constant or slowly time-varying reference inputs with bounded steady error. For the discrete-time case, Cao *et al.* [26] gave a proportional–derivative consensus protocol to track a common time-varying reference input available to only a subset of agents under directed fixed topology. Under some conditions on the varying rate of the reference input, the discretization step-size, and the gain parameter, the consensus error (CE) was shown to be bounded. Zhu and Martínez [29] designed a discrete-time average consensus algorithm for MASs with directed time-varying topologies and different reference inputs. Under a boundness condition on the differences of the reference inputs, it was shown that the steady CE can be made arbitrarily small provided that the discretization step-size is properly chosen.

In this paper, distributed dynamic average consensus problem is investigated for discrete-time MASs under quantized communication data. By exchanging integer-valued information among neighboring agents, we make the MASs track the average of a set of time-varying reference inputs measured independently by each agent. Dynamic consensus protocols suitable for both directed fixed topology and directed time-varying topologies are designed. It is shown that under some boundedness condition on the difference of the reference inputs, the state CE will eventually enter into a small neighborhood of the average of reference inputs that highlights its dependence on the system parameters such as the discretization step-size, the quantization step-size, and the gain parameter. When the number of the agents are fixed, a method to choose the quantization level and the system parameters is also presented.

The remainder of this paper is organized as follows. In Section 2, we present some notions on graph theory and describe the problem to be studied. In Section 3, we devote to discussing the quantized dynamic average consensus problem under both directed fixed topology and directed time-varying topologies. In Section 4, we illustrate the results by two numerical examples. In Section 5, we give some concluding remarks and a discussion on future works.

The following is a table of the basic notations to be used throughout this paper.

I_n	the n dimensional identity matrix.
$\mathbf{1}_n$	an n dimensional vector whose elements are all ones.
$\|X\|_\infty$	the ∞-norm of the matrix X.
$A \odot B$	the Hadamard product of the two matrices A and B.
$\lfloor a \rfloor$	the maximum integer less than or equal to the positive number a.
$\lceil a \rceil$	the minimum integer greater than or equal to the positive number a.
$b \mid_j$	the jth entry of the vector $b \in \mathbb{R}^N$.
$x_i(t)$	the state of agent i.
$\xi_{ji}(t)$	the internal state of the encoder Φ_{ji}.
$\Delta_{ji}(t)$	the output of the encoder Φ_{ji}.
$\hat{x}_{ji}(t)$	the output of the decoder Ψ_{ij}.
h	the gain parameter.
$\mathcal{G}$	the directed communication topology graph.
$\mathcal{L}_\mathcal{G}$	the Laplacian matrix of $\mathcal{G}$.
N_i^+	the in-neighbors of agent i.
N_i^-	the out-neighbors of agent i.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Preliminaries for network modeling

We first give some standard graph modeling for the communication network topology. A weighted digraph (communication graph) $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$ consists a node set $\mathcal{V} = \{1, \cdots, N\}$, an edge set $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A}_\mathcal{G} = [a_{ij}] \in \mathbb{R}^{N \times N}$. A directed edge $(i, j) \in \mathcal{E}_\mathcal{G}$ if and only if there is a communication link from i to j; $a_{ij} \geqslant 0$, and $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}_\mathcal{G}$. When $\mathcal{A}_\mathcal{G}$ is symmetric, $\mathcal{G}$ is called an undirected graph; when $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ for all i, $\mathcal{G}$ is called a balanced graph. A directed path from i_1 to i_k consists of a sequence of edges $(i_1, i_2), (i_2, i_3), \cdots, (i_{k-1}, i_k)$. A directed tree is a digraph, where each node except the root has exactly one parent. The graph is said to be strongly connected if there is a directed path from each node to any other node. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}_\mathcal{G}$. $N_i^+ = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_\mathcal{G}\}$ and $N_i^- = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}_\mathcal{G}\}$ denotes the in-neighbors and out-neighbors of node i, respectively. $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$ denotes the Laplacian matrix of $\mathcal{G}$, where $\mathcal{D}_\mathcal{G} = diag\{\sum_{j \in N_i^+} a_{ij}\}$. For an undirected graph $\mathcal{G}$, the eigenvalues of $\mathcal{L}_\mathcal{G}$ are denoted by $0 = \lambda_1(\mathcal{L}_\mathcal{G}) \leqslant \lambda_2(\mathcal{L}_\mathcal{G}) \leqslant \cdots \leqslant \lambda_N(\mathcal{L}_\mathcal{G})$. The mirror graph of the digraph $\mathcal{G}$ is denoted by $\hat{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}_{\hat{\mathcal{G}}}, \mathcal{A}_{\hat{\mathcal{G}}}\}$, with the same node set, the edge set $\mathcal{E}_{\hat{\mathcal{G}}} = \mathcal{E}_\mathcal{G} \cup \tilde{\mathcal{E}}_\mathcal{G}$ and the symmetric adjacency matrix $\mathcal{A}_{\hat{\mathcal{G}}} = [\hat{a}_{ij}]$, where $\tilde{\mathcal{E}}_\mathcal{G}$ is the reverse edge set of $\mathcal{G}$ obtained by reversing the order of nodes of all the pairs in $\mathcal{E}_\mathcal{G}$, and $\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2}$.

2.2. Problem formulation

For an MAS with N agents, we will consider the distributed dynamic average consensus problem over digital communication channels. The dynamics of each agent is described by the following first-order difference equation:

$$x_i(t + \tau) = x_i(t) + u_i(t), \quad i = 1, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ denote the state and control of the ith agent, respectively; the update time instant t has the form $t = p\tau$ with $p \geqslant 0$ being an integer and $\tau > 0$ being the discretization step-size.

The communication scheme between each pair of adjacent agents consists of a dynamic encoder–decoder pair and an unreliable digital communication channel. Because the digital channel can only transmit symbol information, the real-valued state of each agent should be quantized first at the sender side by using the dynamic encoder, and the symbol information is then decoded into some estimate of the real-valued state at the receiver side by using the dynamic decoder. Because the communication link is unreliable, the communication network topology may be time-varying.

The encoder and decoder are designed based on the following $(2K+1)$-level uniform quantizer $q(\cdot) : \mathbb{R} \to \Lambda = \{0, \pm i, i = 1, \cdots, K\}$:

$$q(y) = \begin{cases} 0, & -1/2 \leqslant y < 1/2, \\ i, & (2i-1)/2 \leqslant y < (2i+1)/2, \\ & i = 1, \cdots, K-1, \\ K, & y \geqslant (2K-1)/2, \\ -q(-y), & y \leqslant -1/2, \end{cases} \tag{2}$$

where Λ denotes the set of quantization levels, and K is a positive integer.

Based on the earlier communication scheme and system dynamics (1), the distributed dynamic average consensus problem is to design a control $u_i(t)$ for each agent i based on only local quantized information such that the states of all the agents converge asymptotically to the average of the reference inputs $\{r_i(t), i = 1, \cdots, N, t \geqslant 0\}$, that is,

$$\lim_{t\to\infty} \left\| x_i(t) - \frac{1}{N} \sum_{i=1}^{N} r_i(t) \right\| = 0,$$

where $r_i(t)$ is the local time-varying reference input of agent i, which can only be known by agent i.

3. DYNAMIC CONSENSUS UNDER QUANTIZED COMMUNICATION DATA

In this section, we will design a distributed dynamic consensus protocol by using quantized data under the directed time-varying topology sequence $\{\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)}\}, t = p\tau, p = 0, 1, \cdots\}$ and obtain the explicit lower bound of quantization levels to ensure the exponential consensus of the whole system.

Intuitively, the main idea for the design of the dynamic consensus protocol is that one part of the protocol is used to drive the state of each agent to track the reference signal measured by itself and, at the same time, the other part of the protocol makes the whole agent achieve consensus under the time-varying topology by using the same error-compensation approach. One difficulty here is to deal with the inconsistence between the internal state of each agent's encoder and the output of its neighbors' decoder caused by the dynamic switches of communication topologies. Thus, the key idea of the dynamic consensus protocol design is to construct a suitable encoder–decoder scheme such that both the sender and the receiver agent can obtain the same estimates of the sender's states even though the communication graph is time-varying, and the error-compensation approach [19] can then be applied to design the consensus protocol. We first present the design of the encoder–decoder scheme and then give the formal statement of the dynamic consensus protocol.

At the sender side of the channel $(j, i) \in \mathcal{E}_{\mathcal{G}}$, agent $j (j = 1, \cdots, N)$ encodes its state by the encoder Φ_{ji} and sends its encoded information to its out-neighbor $i \in N_j^-$. The encoder $\Phi_{ji} \in \Phi_j \triangleq \{\Phi_{ji} : i \in N_j^-\}$ is defined by

$$\begin{aligned} &\xi_{ji}(0) = 0 \\ &\Delta_{ji}(t) = q\left(\epsilon^{-1}(x_j(t) - \xi_{ji}(t-\tau))\right) \\ &\xi_{ji}(t) = \begin{cases} \epsilon \Delta_{ji}(t) + \xi_{ji}(t-\tau), & \text{if } i \text{ receives } \Delta_{ji}(t) \text{ from } j \text{ at time } t, \\ \xi_{ji}(t-\tau), & \text{otherwise}, \quad t = p\tau, \ p = 1, 2, \cdots, \end{cases} \end{aligned} \tag{3}$$

where $\xi_{ji}(t)$ is the internal state of Φ_{ji}, $\Delta_{ji}(t)$ is the output of Φ_{ji}, and $q(\cdot)$ is the uniform quantizer defined in (2) with the quantization step-size ϵ.

At the receiver side of the channel $(j, i) \in \mathcal{E}_{\mathcal{G}}$, agent $i \in N_j^-$ estimates the state of agent j by use of the decoder $\Psi_{ij} \in \Psi_i \triangleq \{\Psi_{ij} : j \in N_i^+\}$, which is defined by

$$\begin{aligned} &\hat{x}_{ji}(0) = 0 \\ &\hat{x}_{ji}(t) = \begin{cases} \epsilon \Delta_{ji}(t) + \hat{x}_{ji}(t-\tau), & \text{if } i \text{ receives } \Delta_{ji}(t) \text{ from } j \text{ at time } t, \\ \hat{x}_{ji}(t-\tau), & \text{otherwise}, \quad t = p\tau, \ p = 1, 2, \cdots, \end{cases} \end{aligned} \tag{4}$$

where $\hat{x}_{ji}(t)$ is the output of Ψ_{ij} at time t. From (3) and (4), the same recursive definition and initial value ensure $\xi_{ji}(t) = \hat{x}_{ji}(t)$. This means the same estimates of the sender's states can be obtained at both the sender side and the receiver side, which is the key to applying the error-compensation design approach.

Remark 1
Here, we do not use the quantizer with the scaling function $g(t)$ as it did in [25] for the quantized static consensus problem and, instead, use a uniform quantizer with a constant quantization step-size ϵ. The main reason is that for the case of dynamic consensus, the steady CE may be non-zero because each agent needs to track the average of time-varying references under quantized data, and in this case, $x_j(t) - \hat{x}_{ji}(t)$ does not converge to 0 as time goes to infinity. Thus, the dynamic encoder–decoder will be saturated if the scaling function $g(t)(g(t) \to 0)$ in [25] is used.

Remark 2
One difficulty in the encoder–decoder pair (3)–(4) is that agent j needs to know whether or not its encoder's output has been received by its out-neighbor $i \in N_j^-$. Similar to [25], we use a noise-free communication feedback channel to let each agent know whether or not its encoder's output is received by its neighbors (see Remark 3.3 in [25]).

Let

$$
\begin{aligned}
r(t) &= [r_1(t), \cdots, r_N(t)]^T, \\
\triangle r_i(t) &= r_i(t) - r_i(t-\tau), \\
\triangle r(t) &= [\triangle r_1(t), \cdots, \triangle r_N(t)]^T, \\
X(t) &= [x_1(t), \cdots, x_N(t)]^T, \\
\delta(t) &= X(t) - J_N X(t), \quad J_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T.
\end{aligned} \tag{5}
$$

Based on the earlier communication scheme, for agent $i (i = 1, \cdots, N)$, we obtain the following consensus protocol over directed time-varying topologies:

$$
\begin{aligned}
u_i(0) &= h \sum_{j \in N_i^+(0)} a_{ij}(0)\hat{x}_{ji}(0) - h \sum_{j \in N_i^-(0)} a_{ji}(0)\xi_{ij}(0) + r_i(0) - x_i(0), \\
u_i(t) &= h \sum_{j \in N_i^+(t)} a_{ij}(t)\hat{x}_{ji}(t) - h \sum_{j \in N_i^-(t)} a_{ji}(t)\xi_{ij}(t) + \triangle r_i(t), \quad t = p\tau, \ p = 1, 2, \cdots,
\end{aligned} \tag{6}
$$

where h is the gain parameter. Substituting (6) into (1) gives the following compact closed-loop system:

$$
X(t+\tau) = \left(I - h\mathcal{L}_{\mathcal{G}(t)}\right) X(t) - h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T\right]\mathbf{1} + \triangle r(t), \tag{7}
$$

where $X(t)$ and $\triangle r(t)$ are given in (5); $Z(t) = [z_{ij}(t)]$, and $z_{ij}(t)$ is defined by

$$
z_{ij}(t) = \begin{cases} \hat{x}_{ji}(t) - x_j(t), & j \in N_i^+, \\ 0, & \text{otherwise.} \end{cases} \tag{8}
$$

Substituting (3) and (7) into (8), we can obtain the following recursive expression for $z_{ij}(t)(i = 1, \cdots, N)$:

$$
z_{ij}(t+\tau) = \begin{cases} M_{ij}^z(t) - \epsilon q\left(\epsilon^{-1} M_{ij}^z(t)\right), & \text{if } j \in N_i^+(t+\tau), \\ M_{ij}^z(t), & \text{if } j \in N_i^+ \setminus N_i^+(t+\tau), \\ 0, & \text{otherwise,} \end{cases} \tag{9}
$$

where $M_{ij}^z(t) = z_{ij}(t) + h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T\right]\mathbf{1}\mid_j + h\mathcal{L}_{\mathcal{G}(t)}\delta(t)\mid_j - \triangle r_j(t)$.

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Remark 3
The dynamic average consensus protocol (6) is inspired by the following views. The first part is based on error-compensation idea [9, 19], whose purpose is to make the states of all agents achieve consensus. The second part is designed for each agent to track the time-varying reference inputs. Specifically, when $\mathcal{G}_t$ is balanced, from $\mathbf{1}^T[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T]\mathbf{1} = 0$, the closed-loop system (7) has the following property:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} x_i(t+\tau) &= \frac{1}{N}\sum_{i=1}^{N} x_i(t) + \frac{1}{N}\sum_{i=1}^{N} \triangle r_i(t) \\ &= \frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N}\sum_{i=1}^{N}\sum_{q=0}^{t/\tau} \triangle r_i(q\tau) = \frac{1}{N}\sum_{i=1}^{N} r_i(t).\end{aligned} \tag{10}$$

Thus, the protocol (6) makes the state average of the MAS consistent with that of the reference inputs. This property is important for the convergence analysis of the closed-loop system (7).

Remark 4
To apply the dynamic protocol (6), each agent needs to know the following information: the reference signal measured by itself, the neighbor link weights, the output of its encoders and its in-neighbors' encoders, and the quantization step-size ϵ. From the discussions later, the choice of the quantization step-size requires the knowledge of the upper bound of the initial states C_x, the upper bound of the difference of the reference signal $\|\triangle r(t)\|_\infty$, the upper bound of the norm of the Laplacian matrices L, and the positive lower bound of the algebraic connectivity of the mirror graphs.

For convenience of the convergence analysis of (7), we make the following assumptions on the time-varying communication graph sequence $\{\mathcal{G}(t), t = p\tau, p = 0, 1, \cdots\}$, and the time-varying reference inputs $\{r_i(t), t \geqslant 0, i = 1, \cdots, N\}$.

(A1) $\{\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)}\}, t = p\tau, p = 0, 1, \cdots\}$ is a balanced digraph sequence, and there is an integer $l_0 > 0$ such that $\inf_{m\geqslant 0} \lambda_{ml_0}^{l_0} \geqslant \lambda_0 > 0$, where $\lambda_k^{l_0} = \lambda_2(\mathcal{L}_{\hat{\mathcal{G}}_k^{l_0}}), \mathcal{G}_k^{l_0} = \sum_{i=k}^{k+\tau_{l_0}-1} \mathcal{G}(k + i\tau), \tau_{l_0} = \frac{l_0}{\tau}, \hat{\mathcal{G}}_k^{l_0}$ is the mirror graph of $\mathcal{G}_k^{l_0}$.
(A2) There is a positive integer p_0, such that for any time instant $t \geqslant 0$ and any agent $j \in N_i^+, i = 1, \cdots, N, j \in N_i^+(t_1)$ holds at least once in $[t, t + p_0\tau)$. In addition, for any time instant t_2 satisfying $t_2 - t \leqslant p_0\tau$, we have $\sup_t \|r(t) - r(t_2)\|_\infty \leqslant C_p$, where C_p is a positive constant.
(A3) For any $\tau > 0$, there is a constant $\theta > 0$ such that

$$\triangle R(t) \triangleq \|\triangle r(t) - J_N \triangle r(t)\|_\infty \leqslant \tau\theta, \quad \forall t \geqslant 0.$$

Remark 5
Assumption (A1) is equivalent to the periodical connectivity condition [17, Lemma 4.1]: there is a positive integer l_0 such that for any $t \geqslant 0, \sum_{k=t}^{t+l_0-1} \mathcal{G}(k)$ contains a spanning tree. From [25, Remark 3.5], Assumption (A1) is a sufficient and necessary condition to ensure an exponential convergence of the distributed consensus protocol for the case of ideal communication data and directed balanced time-varying topology. In the case of finite communication data rate, Assumption (A2) ensures that the encoder (3) and decoder (4) of each agent are not saturated. If the quantization levels of each agent are many enough and ensure that the dynamic encoder and decoder are not saturated, then we can obtain the consensus convergence even without Assumption (A2) but cannot achieve the quantitative relationship between the communication data rate and the associated system parameters. When $\tau \to 0$, Assumption (A3) becomes $\|\dot{r}_i(t) - J_N\dot{r}(t)\|_\infty \leqslant \theta$, and thus, Assumption (A3) can be regarded as a discrete-time counterpart of $\|\dot{r}_i(t) - J_N\dot{r}(t)\|_\infty \leqslant \theta$ for some fixed θ and all time instants t. In the dynamic consensus works with ideal communication channel, similar assumptions include $\| \triangle r(t)/\tau\| \leqslant \theta$ (bounded changing-rate condition [26]) and $\| \triangle r_{\max}(t) - \triangle r_{\min}(t)\| \leqslant \tau\theta$ (relatively bounded first-order difference condition [29]) with $\triangle r_{\max}(t) = \max_{i\in\mathcal{V}} \triangle r_i(t), \triangle r_{\min}(t) = \min_{i\in\mathcal{V}} \triangle r_i(t)$.

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We now consider the convergence analysis of the closed-loop system (7) and give a lower bound on the communication bits required by each pair of adjacent agents at each time step to guarantee the dynamic average consensus.

Theorem 1
For the systems (7), (9), (3), and (4) under the time-varying communication topology sequence $\{\mathcal{G}(t), t = p\tau, p = 0, 1, \cdots\}$, suppose Assumptions (A1), (A2), and (A3) hold. Let $d^* \geqslant \sup_k d^*(k)$, where $d^*(k)$ is the degree of $\mathcal{G}(k)$, that is, $d^*(k) = \max_{i\in\mathcal{V}} \sum_{j=1}^{N} a_{ij}$:

$$
\begin{aligned}
K_1(h,\tau,\epsilon,\epsilon_1,\epsilon_2) &= \max\left\{\left\lfloor \epsilon^{-1}\left((1+2hd^*)C_x + 2hd^*C_\delta + C_r\right) - \frac{1}{2}\right\rfloor + 1, \left\lfloor M_1 - \frac{1}{2}\right\rfloor + \frac{3}{2}\right\},\\
M_1 = M_1(h,\tau,\epsilon,\epsilon_1,\epsilon_2) &= \epsilon^{-1}C_p M_2^{\frac{1}{2}} + \epsilon^{-1}C_r + hd^* + 2h\epsilon^{-1}d^*M_2^{\frac{1}{2}},\\
M_2 = NC_\delta^2\rho_{h,\epsilon_2}^{2\tau_{l_0}} &+ N\epsilon_2^{-1}(2hd^*C_x+\tau\theta)^2\rho_{h,\epsilon_2}^{2\tau_{l_0}+1} + \frac{N\epsilon_2^{-1}(h\epsilon d^*+\tau\theta)^2\rho_{h,\epsilon_2}^{\tau_{l_0}+1}\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1-\rho_{h,\epsilon_2}}\\
&+ \frac{N\rho_{h,\epsilon_1}\epsilon_1^{-1}(hd^*\epsilon+\tau\theta)^2\rho_{h,\epsilon_2}^{\tau_{l_0}}}{1-\rho_{h,\epsilon_1}}\sum_{j=0}^{2\tau_{l_0}-2}\left(\tau_{l_0} - |j-(\tau_{l_0}-1)|\right)\sum_{l=0}^{j} C_j^l h^l L^l\\
&+ \frac{N\rho_{h,\epsilon_2}\epsilon_2^{-1}(h\epsilon d^*+\tau\theta)^2\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1-\rho_{h,\epsilon_2}},\\
\Theta_1 &= \left\{\frac{\rho_{h,\epsilon_1}\epsilon_1^{-1}\rho_{h,\epsilon_2}^{\tau_{l_0}}}{1-\rho_{h,\epsilon_1}}\sum_{j=0}^{2\tau_{l_0}-2}\left(\tau_{l_0}-|j-(\tau_{l_0}-1)|\right)\sum_{l=0}^{j}C_j^l h^l L^l + \frac{\rho_{h,\epsilon_2}\epsilon_2^{-1}\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1-\rho_{h,\epsilon_2}}\right\}^{\frac{1}{2}},\\
\rho_{h,\epsilon_1} &= 1 - 2h\lambda_0 + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l L^l + \epsilon_1, \quad \rho_{h,\epsilon_2} = 1 + 2hL + h^2L^2 + \epsilon_2,
\end{aligned}
\tag{11}
$$

and $\tau_{l_0} = \frac{l_0}{\tau}$; C_x, C_δ and C_r are constants satisfying $C_x \geqslant \|X(0)\|_\infty$, $C_\delta \geqslant \|X(0) - J_N X(0)\|_\infty$ and $C_r \geqslant \sup_{t\geqslant 0}\|\triangle r(t)\|_\infty$; L is a constant satisfying $L \geqslant \sup_k \|\mathcal{L}_{\mathcal{G}(k)}\|$; C_j^l denotes the combinatorial number by choosing l numbers from j numbers. For any given $\bar{\vartheta} > 0$, and $K \geqslant K_1(h,\tau,\epsilon,\epsilon_1,\epsilon_2)$, choose the positive parameter vector $(h,\tau,\epsilon,\epsilon_1,\epsilon_2)$ such that $\rho_{h,\epsilon_1} \in (0,1)$ and

$$
hd^*\epsilon + \tau\theta \leqslant \frac{\bar{\vartheta}}{\sqrt{N}\Theta_1}, \tag{12}
$$

then under the $(2K+1)$-level uniform quantizer (2) with quantization step-size ϵ, the system (7) achieves the dynamic average consensus with a steady state error ϑ bounded by $\bar{\vartheta}$, that is,

$$
\vartheta = \max_{i\in\mathcal{V}} \limsup_{t\to\infty} \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} r_j(t-\tau)\right| \leqslant \bar{\vartheta}. \tag{13}
$$

Proof
See Appendix A. □

Remark 6
On the choice of the parameters $(h,\tau,\epsilon,\epsilon_1,\epsilon_2)$, we need to consider the following factors: the number of quantization levels $2K+1$ should be as less as possible, the discretization step-size should be as large as possible, and the number ρ_{h,ϵ_1} measuring the convergence rate should be as small as possible. Unfortunately, these requirements contradicts each other. Take the relationship between ϵ and τ for example, when h and ϵ_1 are fixed, by (12) and (11), we know that in order to make the number of quantization levels $2K+1$ small, ϵ should be large. But from (11), it is hard to make

ϵ and τ large at the same time. Thus, there should be a trade-off. By sacrificing some convergence rate of the closed-loop system, we can decrease the number of quantization levels through solving the following minimization problem:

$$\min_{(h,\tau,\epsilon,\epsilon_1,\epsilon_2)\in\mathcal{S}_1} \max\left\{\left\lfloor \epsilon^{-1}\left((1+2hd^*)C_x + 2hd^*C_\delta + C_r\right) - \frac{1}{2}\right\rfloor + 1, \left\lfloor M_1 - \frac{1}{2}\right\rfloor + \frac{3}{2}\right\},$$

where

$$\mathcal{S}_1 = \left\{(h,\tau,\epsilon,\epsilon_1,\epsilon_2) : h>0, \epsilon>0, \tau>0, \epsilon_1>0, \epsilon_2>0, \rho_{h,\epsilon_1}\in(0,1), hd^*\epsilon + \tau\theta \leqslant \frac{\bar{\vartheta}}{\sqrt{N}\Theta_1}\right\}$$

with $\rho_{h,\epsilon_1}, \rho_{h,\epsilon_2}, M_1$ and Θ_1 being defined in (11).

Based on the earlier discussions for the directed time-varying topology, the special case under the directed fixed topology $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$ can be easily obtained. The main idea of the protocol design is the same as the time-varying topology case. Thus, later, we just present results about the communication scheme, distributed control, and the convergence results, illustrating the differences from the time-varying topology case. The following assumption on the fixed communication graph $\mathcal{G}$ is needed:

(A4) $\mathcal{G}$ is directed, balanced, and contains a spanning tree.

The encoder–decoder scheme suitable for the noise-free digital channel $(j,i)\in\mathcal{E}_\mathcal{G}$ contains an encoder Φ_j at the sender side of agent j:

$$\begin{cases} \xi_j(0) = 0, \\ \Delta_j(t) = q\left(\epsilon^{-1}(x_j(t) - \xi_j(t-\tau))\right), \\ \xi_j(t) = \epsilon\Delta_j(t) + \xi_j(t-\tau), \quad t = p\tau,\ p = 1,2,\cdots, \end{cases} \tag{14}$$

and the decoder Ψ_i at the receiver side of agent i $\left(i\in N_j^-\right)$:

$$\begin{cases} \hat{x}_{ji}(0) = 0, \\ \hat{x}_{ji}(t) = \epsilon\Delta_j(t) + \hat{x}_{ji}(t-\tau), \quad t = p\tau,\ p = 1,2,\cdots, \end{cases} \tag{15}$$

where $\xi_j(t)$ is the internal state of Φ_j, $\Delta_j(t)$ is the output of Φ_j, $\hat{x}_{ji}(t)$ is the output of Ψ_i, and $q(\cdot)$ is the uniform quantizer defined in (2) with the quantization step-size ϵ.

For the directed fixed topology case, the error-compensation type average consensus protocol (6) becomes

$$u_i(t) = \sum_{j\in N_i^+} a_{ij}\left(\hat{x}_{ji}(t) - \xi_i(t)\right) + \triangle r_i(t), \quad t = p\tau,\ p = 0,1,\cdots. \tag{16}$$

This together with (14), (15), and (1) gives the closed-loop system in the following compact form:

$$\begin{cases} X(t+\tau) = (I - h\mathcal{L}_\mathcal{G})X(t) + h\mathcal{L}_\mathcal{G}e(t) + \triangle r(t), \\ \hat{X}(t+\tau) = \epsilon Q\left[\epsilon^{-1}(X(t+\tau) - \hat{X}(t))\right] + \hat{X}(t), \end{cases} \tag{17}$$

where $X(t)$ is defined in (5), $\hat{X}(t) = [\xi_1(t),\cdots,\xi_N(t)]^T, e(t) = X(t) - \hat{X}(t)$, $Q\left([y_1,\cdots,y_N]^T\right) = [q(y_1),\cdots,q(y_N)]^T$.

The consensus results corresponding to Theorem 1 can be summarized into the following theorem.

Theorem 2

For system (17), assume Assumptions (A3) and (A4) hold. Let

$$K_2(h,\tau,\epsilon,\epsilon_3) = \max\left\{\left\lfloor \epsilon^{-1}(C_x + C_r) - \frac{1}{2}\right\rfloor + 1, \lfloor\Theta_2\rfloor + 1\right\},$$

$$\Theta_2 = hd^* + \epsilon^{-1}C_r + h\epsilon^{-1}\rho_{h,\epsilon_3}^{\frac{1}{2}}\sqrt{N}\overline{L}\left\{C_\delta^2 + \epsilon_3^{-1}(1-\rho_{h,\epsilon_3})^{-1} \cdot \left(h\epsilon\overline{L}/2 + \tau\theta\right)^2 + \epsilon_3^{-1}\left(hC_x\overline{L} + \tau\theta\right)^2\right\}^{\frac{1}{2}}, \tag{18}$$

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and C_x, C_δ and C_r are constants satisfying $C_x \geqslant \|X(0)\|_\infty, C_\delta \geqslant \|X(0) - J_N X(0)\|_\infty$ and $C_r \geqslant \sup_{t\geqslant 0} \| \triangle r(t)\|_\infty$; $\bar{\lambda}_0, \overline{L}$ are positive constants, satisfying $0 < \bar{\lambda}_0 \leqslant \lambda_2(\mathcal{L}_{\hat{\mathcal{G}}}), \overline{L} \geqslant \|\mathcal{L}_{\mathcal{G}}\|$; $\hat{\mathcal{G}}$ is the mirror graph of $\mathcal{G}$. For any given $\bar{\vartheta} > 0$, and $K \geqslant K_2(h, \tau, \epsilon, \epsilon_3)$, choose the positive parameter vector $(h, \tau, \epsilon, \epsilon_3)$ properly such that

$$\begin{aligned} &\rho_{h,\epsilon_3} \triangleq 1 - 2h\bar{\lambda}_0 + h^2\overline{L}^2 + \epsilon_3 \in (0,1), \\ &\frac{h\overline{L}}{2}\epsilon + \tau\theta \leqslant \frac{\bar{\vartheta}}{\sqrt{N}}\left[\epsilon_3\left(\rho_{h,\epsilon_3}^{-1} - 1\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Then, under the $(2K+1)$-level uniform quantizer with the quantization step-size ϵ, the system (17) achieves dynamic average consensus with a steady state error satisfying

$$\vartheta \triangleq \max_{i\in\mathcal{V}} \limsup_{t\to\infty} \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} r_j(t-\tau)\right| \leqslant \bar{\vartheta}. \tag{20}$$

Proof
See Appendix B. □

Remark 7
Similar to the time-varying topology case, the choice of the parameter vector $(h, \tau, \epsilon, \epsilon_3)$ should give an overall consideration of factors such as quantization levels, discretization step-size and convergence rate, depending on the implementation situation of the protocol (14)–(16). As stated in Remark 6, if some convergence rate can be sacrificed, then one can decrease the quantization level by solving the following minimization problem:

$$\min_{h,\tau,\epsilon,\epsilon_3\in\mathcal{S}_2} \max\left\{\left\lfloor \epsilon^{-1}(C_x + C_r) - \frac{1}{2}\right\rfloor + 1, \lfloor\Theta_2\rfloor + 1\right\},$$

where

$$\mathcal{S}_2 = \left\{(h,\tau,\epsilon,\epsilon_3) : h>0, \epsilon>0, \tau>0, \epsilon_3>0, \rho_{h,\epsilon_3} \in (0,1), \frac{h\overline{L}}{2}\epsilon + \tau\theta \leqslant \frac{\bar{\vartheta}}{\sqrt{N}}\left[\epsilon_3\left(\rho_{h,\epsilon_3}^{-1} - 1\right)\right]^{\frac{1}{2}}\right\}$$

with ρ_{h,ϵ_3} and Θ_2 defined in (19) and (18), respectively.

4. NUMERICAL EXAMPLES

In this section, we will give two numerical examples to illustrate the results of Section 3.

Example 1
Consider the network of three agents with the directed communication graph $\mathcal{G} = \{\mathcal{V} = \{1,2,3\}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}} = [a_{ij}]_{3\times 3}\}$, where $\mathcal{E}_{\mathcal{G}} = \{(1,2),(2,1),(1,3),(3,1)\}, a_{12} = 0.8, a_{21} = a_{31} = a_{23} = 0.4$, and $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}_{\mathcal{G}}$. The reference inputs of each agent are given by $r_1(t) = 4\sin t + 0.5t - 5$, $r_2(t) = 8\sin t + 0.5t + 1$ and $r_3(t) = 10\cos t + t + 5$. Set $\bar{\vartheta} = 0.1$. Then, by Theorem 2 we can choose $h = 0.04, \tau = 5\times 10^{-4}, \epsilon = 0.0419$, and $\epsilon_3 = 0.02$ to make (19) hold. Thus, the lower bound of the number of quantization levels is $K = 359$. Applying the consensus protocol (14)–(16) to the system (1), the evolution curves of agents' states, references, and the average of references are shown in Figure 1, and the curve of the CE of each agent is shown in Figure 2. It can be seen that the steady CE of each agent is bounded by $\bar{\vartheta}$.

Example 2
Consider the network of three agents with the time-varying communication graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)} = [a_{ij}(t)]_{3\times 3}\}$, where $\mathcal{E}_{\mathcal{G}(t)} = \{(1,2),(2,1)\}, a_{12}(t) = a_{21}(t) = 0.8, a_{ij}(t) = 0$ if $(i,j) \notin \mathcal{E}_{\mathcal{G}(t)}$ when $t = 2k, k = 0,1,\cdots$; $\mathcal{E}_{\mathcal{G}(t)} = \{(1,3),(3,1)\}, a_{13}(t) = a_{31}(t) = 0.8, a_{ij}(t) = 0$ if $(i,j) \notin \mathcal{E}_{\mathcal{G}(t)}$ when $t = 2k+1, k = 0,1,\cdots$. It can be seen that $\mathcal{G}(t)$ is balanced and

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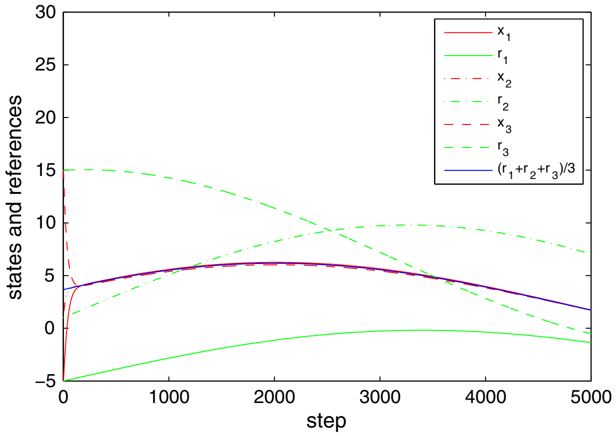

Figure 1. Curves of agents' states, references, and average of references under directed fixed topology.

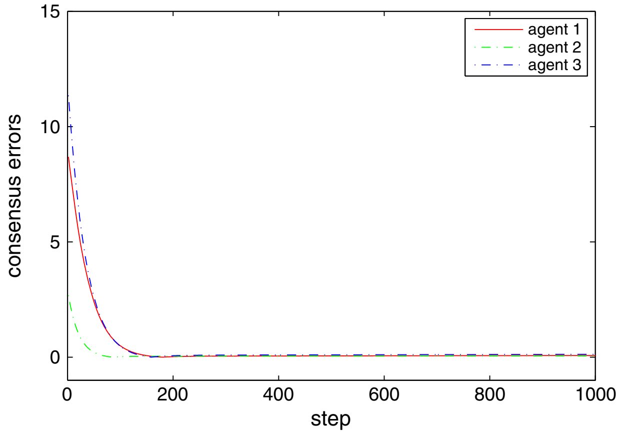

Figure 2. Curves of consensus errors under directed fixed topology.

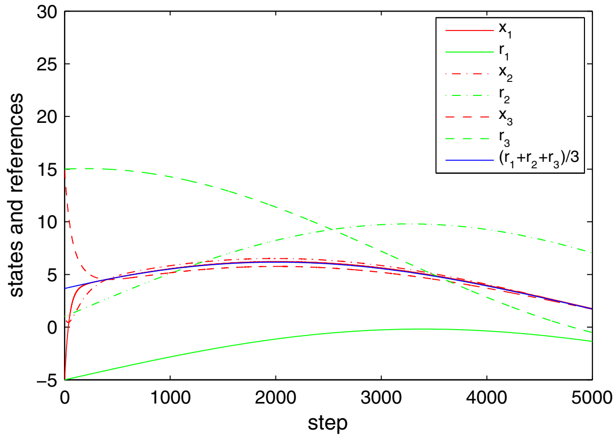

Figure 3. Curves of agents' states, references, and average of references under directed time-varying topologies.

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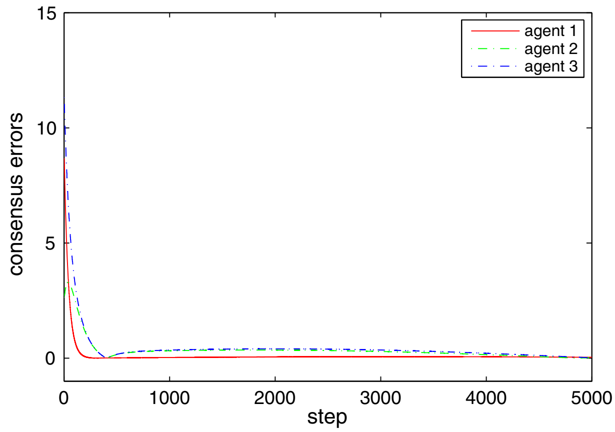

Figure 4. Curves of consensus errors under directed time-varying topologies.

$\mathcal{G}_t^2 = \sum_{i=t}^{t+1} \mathcal{G}(i)$ has a spanning tree. The reference inputs of each agent are given as in Example 1. Set $\bar{\vartheta} = 0.1$. Then, by Theorem 1, we can choose $h = 0.02, \tau = 5 \times 10^{-4}, \epsilon = 0.0161, \epsilon_1 = 0.01$ and $\epsilon_2 = 0.01$ to make $\rho_{h,\epsilon_1} \in (0, 1)$ and (12) hold. Thus, the lower bound of the number of quantization levels is $K = 983$. Applying the consensus protocol (3)–(6) to the system (1), the curves of agents' states, references, and the average of references are shown in Figure 3, and the CE of each agent is shown in Figure 4. It can be seen that the steady CE of each agent is bounded by $\bar{\vartheta}$.

5. CONCLUSIONS

This paper has considered the distributed dynamic average consensus of MASs with noise-free digital communication channels. To deal with the saturation caused by the dynamic CE, we use a uniform quantizer with constant quantization step to design the encoder–decoder scheme. Distributed quantized dynamic consensus protocols suitable for the time-varying topologies are developed, which are shown to be able to ensure the states of all the agents achieve dynamic consensus with arbitrarily small error by properly choosing system parameters. The lower bound of the required quantization levels and the method to choose system parameters are also presented. Future research topics include the quantized dynamic consensus for higher order MASs over random switching topologies and noisy digital communication channels.

APPENDIX A: PROOF OF THEOREM 1

Proof
From (10), $\mathcal{L}_{\mathcal{G}(t)} J_N = 0$, and (7), we have

$$\delta(t+\tau) = (I - h\mathcal{L}_{\mathcal{G}(t)})\delta(t) - h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T\right]\mathbf{1} + \triangle r(t) - J_N \triangle r(t), \quad \text{(A.1)}$$

where $\delta(t)$, $\triangle r(t)$ are given by (5), $Z(t)$ has the recursive form of (9).

We now show that no quantizer is saturated. From (3) and (4), we have $\hat{x}_{ij}(0) = 0, j \in N_i^+, i = 1, \cdots, N$. By $K \geqslant K_1(h, \tau, \epsilon, \epsilon_1, \epsilon_2)$ and (11), we know that

$$\begin{aligned}
\epsilon^{-1}|M_{ij}^z(0)| &\leqslant \epsilon^{-1}\left(\|X(0)\|_\infty + h\|\mathcal{L}_{\mathcal{G}(0)}\|_\infty \cdot \|\delta(0)\|_\infty + h\|(\mathcal{L}_{\mathcal{G}(0)} \odot Z(0))\mathbf{1}\|_\infty\right. \\
&\quad \left. + h\|(\mathcal{L}_{\mathcal{G}(0)} \odot Z(0))^T\mathbf{1}\|_\infty + \|\triangle r(0)\|_\infty\right) \\
&< \left\lfloor \epsilon^{-1}\left(C_x + 2hd^*C_\delta + 2hd^*C_x + C_r\right) - \frac{1}{2} \right\rfloor + \frac{3}{2} \\
&\leqslant K + \frac{1}{2}, \quad j \in N_i^+, \quad i = 1, \cdots, N.
\end{aligned}$$

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Thus, no quantizer is saturated at the initial time. Suppose that at time $k = 0, \tau, \cdots, t$, no quantizer is saturated. Then, we can claim that no quantizer is saturated at time $t + \tau$. By direct computations, one can obtain $\mathcal{L}_{\mathcal{G}(k)} \odot Z(k) = \bar{Z}(k) = [\bar{z}_{ij}(k)]$, where

$$\bar{z}_{ij}(k) = \begin{cases} a_{ij}(k) z_{ij}(k), & \text{if } j \in N_i^+(k), \\ 0, & \text{otherwise}, \quad k = \tau, \cdots, t + \tau. \end{cases}$$

Thus, by (9), we have $|\bar{z}_{ij}(k)| \leqslant \frac{a_{ij}(k)\epsilon}{2}$, and

$$\left\| (\mathcal{L}_{\mathcal{G}(k)} \odot Z(k)) \mathbf{1} \right\|_\infty \leqslant \frac{d^* \epsilon}{2}, \quad \left\| (\mathcal{L}_{\mathcal{G}(k)} \odot Z(k))^T \mathbf{1} \right\|_\infty \leqslant \frac{d^* \epsilon}{2}. \tag{A.2}$$

For any positive integer m, by (A.1), we have

$$\begin{aligned} \delta((m+1)l_0) = {} & \Phi((m+1)l_0, ml_0)\delta(ml_0) + \sum_{j=0}^{\tau_{l_0}-1} \Phi((m+1)l_0 - \tau, ml_0 + j\tau) \\ & \cdot \left[-h(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau))\mathbf{1} + h(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau))^T \mathbf{1} \right. \\ & \left. + \triangle r(ml_0 + j\tau) - J_N \triangle r(ml_0 + j\tau) \right], \end{aligned} \tag{A.3}$$

where $\Phi(n + \tau, i) = (I - h\mathcal{L}_{\mathcal{G}(n)})\Phi(n, i)$, $\Phi(i, i) = I$. Thus,

$$\begin{aligned} \|\delta((m+1)l_0)\|^2 = {} & \delta^T(ml_0)\Phi^T((m+1)l_0, ml_0)\Phi((m+1)l_0, ml_0)\delta(ml_0) \\ & + 2\delta^T(ml_0)\Phi^T((m+1)l_0, ml_0) \sum_{j=0}^{\tau_{l_0}-1} \Phi((m+1)l_0 - \tau, ml_0 + j\tau) \\ & \left[h(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau))^T \mathbf{1} - h(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau))\mathbf{1} \right. \\ & \left. + \triangle r(ml_0 + j\tau) - J_N \triangle r(ml_0 + j\tau) \right] + I_{ml_0}^\tau \\ \triangleq {} & I_3 + I_4 + I_{ml_0}^\tau, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} I_{ml_0}^\tau = {} & \sum_{j=0}^{\tau_{l_0}-1} \left[h\mathbf{1}^T(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau)) - h\mathbf{1}^T(\mathcal{L}_{\mathcal{G}(ml_0+j\tau)} \odot Z(ml_0 + j\tau))^T \right. \\ & \left. + \triangle r^T(ml_0 + j\tau) - \triangle r^T(ml_0 + j\tau)J_N \right] \Phi^T((m+1)l_0 - \tau, ml_0 + j\tau) \\ & \cdot \sum_{k=0}^{\tau_{l_0}-1} \Phi((m+1)l_0 - \tau, ml_0 + k\tau) \left[h(\mathcal{L}_{\mathcal{G}(ml_0+k\tau)} \odot Z(ml_0 + k\tau))^T \mathbf{1} \right. \\ & \left. - h(\mathcal{L}_{\mathcal{G}(ml_0+k\tau)} \odot Z(ml_0 + k\tau))\mathbf{1} + \triangle r(ml_0 + k\tau) - J_N \triangle r(ml_0 + k\tau) \right]. \end{aligned}$$

By Assumption (A1), we have

$$\begin{aligned} \left\| \Phi^T((m+1)l_0, ml_0)\Phi((m+1)l_0, ml_0) \right\| & \leqslant \left\| I - 2h \sum_{i=0}^{\tau_{l_0}-1} \mathcal{L}_{\hat{\mathcal{G}}(ml_0+i\tau)} \right\| + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l \left(\sup_{t \geqslant 0} \|\mathcal{L}_{\mathcal{G}(t)}\| \right)^l \\ & \leqslant 1 - 2h\lambda_0 + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l L^l. \end{aligned} \tag{A.5}$$

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For any $\epsilon_1 > 0$, by $2x^T y \leqslant \epsilon_1 x^T x + \epsilon_1^{-1} y^T y, \forall x, y \in \mathbb{R}^N$, we have

$$I_4 \leqslant \epsilon_1 \|\delta(ml_0)\|^2 + \epsilon_1^{-1}\left(1 - 2h\lambda_0 + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l L^l\right) \cdot I_{ml_0}^{\tau}, \tag{A.6}$$

where I_4 and $I_{ml_0}^{\tau}$ are defined in (A.4). Noticing (A.2), Assumption (A3), and by direct computations, we can obtain

$$I_{ml_0}^{\tau} \leqslant N(hd^*\epsilon + \tau\theta)^2 \sum_{j=0}^{2\tau_{l_0}-2} (\tau_{l_0} - |j - (\tau_{l_0} - 1)|) \sum_{l=0}^{j} h^l C_j^l L^l, \quad m \geqslant 1.$$

This together with (A.4)–(A.6) implies that

$$\begin{aligned}
\|\delta((m+1)l_0)\|^2 &\leqslant \rho_{h,\epsilon_1}\|\delta(ml_0)\|^2 + \left[1 + \epsilon_1^{-1}\left(1 - 2h\lambda_0 + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l L^l\right)\right] \cdot I_{ml_0}^{\tau} \\
&\leqslant \rho_{h,\epsilon_1}^m \|\delta(l_0)\|^2 + N(hd^*\epsilon + \tau\theta)^2 \left[1 + \epsilon_1^{-1}\left(1 - 2h\lambda_0 + \sum_{l=2}^{2\tau_{l_0}} h^l C_{2\tau_{l_0}}^l L^l\right)\right] \\
&\quad \cdot \sum_{j=0}^{2\tau_{l_0}-2} (\tau_{l_0} - |j - (\tau_{l_0} - 1)|) \sum_{l=0}^{j} C_j^l h^l L^l \cdot \left(\frac{1 - (\rho_{h,\epsilon_1})^m}{1 - \rho_{h,\epsilon_1}}\right),
\end{aligned} \tag{A.7}$$

where ρ_{h,ϵ_1} is defined in (11). In addition, by (A.1), we have

$$\begin{aligned}
\|\delta(t+\tau)\|^2 &\leqslant \left(1 + 2h \sup_{k\geqslant 0}\|\mathcal{L}_{\mathcal{G}(k)}\| + h^2 \sup_{k\geqslant 0}\|\mathcal{L}_{\mathcal{G}(k)}\|^2 + \epsilon_2\right)\|\delta(t)\|^2 \\
&\quad + \left[\epsilon_2^{-1}\left(1 + 2h \sup_{k\geqslant 0}\|\mathcal{L}_{\mathcal{G}(k)}\| + h^2 \sup_{k\geqslant 0}\|\mathcal{L}_{\mathcal{G}(k)}\|^2\right) + 1\right] \cdot \|h(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T \mathbf{1} \\
&\quad - h(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))\mathbf{1} + \triangle r(t) - J_N \triangle r(t)\|^2 \\
&\leqslant \rho_{h,\epsilon_2}\|\delta(t)\|^2 + \left[\epsilon_2^{-1}(1 + 2hL + h^2L^2) + 1\right] \cdot \|h(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T \mathbf{1} \\
&\quad - h(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))\mathbf{1} + \triangle r(t) - J_N \triangle r(t)\|^2,
\end{aligned} \tag{A.8}$$

where ρ_{h,ϵ_2} is defined in (11). This together with (A.2), $\|(\mathcal{L}_{\mathcal{G}(0)} \odot Z(0))\mathbf{1}\|_\infty \leqslant d^* C_x$ and (A.8) gives

$$\begin{aligned}
\|\delta(l_0)\|^2 \leqslant\; & NC_\delta^2 \rho_{h,\epsilon_2}^{\tau_{l_0}} + N\left(2hd^*C_x + \tau\theta\right)^2 \left[\epsilon_2^{-1}(1 + 2hL + h^2L^2) + 1\right]\rho_{h,\epsilon_2}^{\tau_{l_0}} \\
& + \frac{N(h\epsilon d^* + \tau\theta)^2\left(1 - \rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1 - \rho_{h,\epsilon_2}}\left[\epsilon_2^{-1}(1 + 2hL + h^2L^2) + 1\right].
\end{aligned} \tag{A.9}$$

For any $t \geqslant 0$, set $m_t = \lfloor \frac{t}{l_0} \rfloor$. Then, $0 \leqslant t - m_t l_0 \leqslant l_0$. By (A.7), (A.8), (A.9), and $\rho_{h,\epsilon_1} \in (0,1)$, we have

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$$
\begin{aligned}
\|\delta(t+\tau)\|^2 \leqslant{} & \rho_{h,\epsilon_2}^{(t+\tau-m_t l_0)/\tau}\|\delta(m_t l_0)\|^2 + \sum_{i=0}^{(t-m_t l_0)/\tau} \rho_{h,\epsilon_2}^{i}\left[\epsilon_2^{-1}(1+2hL+h^2L^2)+1\right] \\
& \cdot\left\|h(\mathcal{L}_{\mathcal{G}(t-i\tau)}\odot Z(t-i\tau))\mathbf{1}-h\left(\mathcal{L}_{\mathcal{G}(t-i\tau)}\odot Z(t-i\tau)\right)^T\mathbf{1}+\triangle r(t-i\tau)-J_N\triangle r(t-i\tau)\right\|^2 \\
\leqslant{} & \left\{NC_\delta^2\rho_{h,\epsilon_2}^{2\tau_{l_0}} + N\epsilon_2^{-1}(2hd^*C_x+\tau\theta)^2\rho_{h,\epsilon_2}^{2\tau_{l_0}+1}\right. \\
& \left.+\frac{N\epsilon_2^{-1}(h\epsilon d^*+\tau\theta)^2\rho_{h,\epsilon_2}^{\tau_{l_0}+1}\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1-\rho_{h,\epsilon_2}}\right\}\rho_{h,\epsilon_1}^{m_t-1} \\
& + N\rho_{h,\epsilon_1}\epsilon_1^{-1}(hd^*\epsilon+\tau\theta)^2\rho_{h,\epsilon_2}^{\tau_{l_0}}\sum_{j=0}^{2\tau_{l_0}-2}(\tau_{l_0}-|j-(\tau_{l_0}-1)|)\sum_{l=0}^{j}C_j^l h^l L^l\frac{1-\rho_{h,\epsilon_1}^{m_t-1}}{1-\rho_{h,\epsilon_1}} \\
& + \frac{N\rho_{h,\epsilon_2}\epsilon_2^{-1}(h\epsilon d^*+\tau\theta)^2\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{1-\rho_{h,\epsilon_2}}.
\end{aligned}
\tag{A.10}
$$

By the definition of $z_{ij}(t+\tau)$ in (9) and Assumption (A2), we have

$$
\left|z_{ij}(t+\tau)\right| \leqslant \max\left\{\frac{\epsilon}{2}, \left|\hat{x}_{ji}\left(\tau_{ji}^t\right)-x_j(t+1)\right|\right\}, \tag{A.11}
$$

where $\tau_{ji}^t = \max\{t_1 \leqslant t : j \in N_i^+(t_1)\}$ and $t-\tau_{ji}^t \leqslant p_0\tau$. Furthermore, by the definition of the decoder Ψ_{ji} in (3), we have

$$
\left|\hat{x}_{ji}\left(\tau_{ji}^t\right)-x_j\left(\tau_{ji}^t\right)\right| < \frac{\epsilon}{2}. \tag{A.12}
$$

Similar to (A.10) for $\delta(\tau_{ji}^t)$, and noticing (10), we have

$$
\left|x_j(t+\tau)-J_N r(t)\right| \leqslant M_2^{\frac{1}{2}}, \quad \left|x_j\left(\tau_{ji}^t\right)-J_N r\left(\tau_{ji}^t-1\right)\right| \leqslant M_2^{\frac{1}{2}},
$$

where M_2 is defined by (11). Thus, $\left|x_j(t+\tau)-x_j\left(\tau_{ji}^t\right)\right| \leqslant M_2^{\frac{1}{2}}\left|J_N\left(r(t)-r\left(\tau_{ji}^t-1\right)\right)\right|$. This together with (A.12) gives

$$
\begin{aligned}
\left|\hat{x}_{ji}\left(\tau_{ji}^t\right)-x_j(t+\tau)\right| &= \left|\left(\hat{x}_{ji}\left(\tau_{ji}^t\right)-x_j\left(\tau_{ji}^t\right)\right)-\left(x_j(t+1)-x_j\left(\tau_{ji}^t\right)\right)\right| \\
&\leqslant \frac{\epsilon}{2}+M_2^{\frac{1}{2}}\left|J_N\left(r(t)-r\left(\tau_{ji}^t-1\right)\right)\right|.
\end{aligned}
$$

Therefore, by (11), (A.2), and (A.11), one can obtain

$$
\begin{aligned}
\epsilon^{-1}|M_{ij}^z(t+\tau)| \leqslant{} & \epsilon^{-1}|z_{ij}(t+\tau)|+h\epsilon^{-1}\left\|\left[(\mathcal{L}_{\mathcal{G}(t+\tau)}\odot Z(t+\tau))-(\mathcal{L}_{\mathcal{G}(t+\tau)}\odot Z(t+\tau))^T\right]\mathbf{1}\right\|_\infty \\
& + h\epsilon^{-1}\|\mathcal{L}_{\mathcal{G}(t+\tau)}\delta(t+\tau)\|_\infty+\epsilon^{-1}\|\triangle r(t+\tau)\|_\infty \\
\leqslant{} & \frac{1}{2}+\epsilon^{-1}C_pM_2^{\frac{1}{2}}+\epsilon^{-1}C_r+hd^*+2h\epsilon^{-1}d^*M_2^{\frac{1}{2}} \\
={} & \frac{1}{2}+M_1(h,\tau,\epsilon,\epsilon_1,\epsilon_2) < \left\lfloor M_1(h,\tau,\epsilon,\epsilon_1,\epsilon_2)-\frac{1}{2}\right\rfloor+2 \leqslant K+\frac{1}{2},
\end{aligned}
$$

where M_2 is defined in (11). Thus, no quantizer is saturated at time $t + \tau$. From (10) and (A.10), it follows that the dynamic consensus has a steady error ϑ as $t \to \infty$, which is

$$\begin{aligned}\vartheta(h,\tau,\epsilon,\epsilon_3) &\triangleq \max_{i\in\mathcal{V}} \limsup_{t\to\infty} \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} r_j(t-\tau)\right| = \max_{i\in\mathcal{V}} \limsup_{t\to\infty} \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t)\right| \\ &\leqslant \sqrt{N}\big(hd^*\epsilon+\tau\theta\big)\left\{\frac{\rho_{h,\epsilon_1}\rho_{h,\epsilon_2}^{\tau_{l_0}}}{\epsilon_1(1-\rho_{h,\epsilon_1})}\sum_{j=0}^{2\tau_{l_0}-2}(\tau_{l_0}-|j-(\tau_{l_0}-1)|)\sum_{l=0}^{j} C_j^l h^l L^l \right. \\ &\qquad \left. + \frac{\rho_{h,\epsilon_2}\left(1-\rho_{h,\epsilon_2}^{\tau_{l_0}}\right)}{\epsilon_2(1-\rho_{h,\epsilon_2})}\right\}^{\frac{1}{2}}.\end{aligned}$$

This together with (12) gives (13). □

APPENDIX B: PROOF OF THEOREM 2

Proof

Noticing $\mathbf{1}^T\mathcal{L}_{\mathcal{G}} = \mathcal{L}_{\mathcal{G}}\mathbf{1} = 0$, from (17), we have $J_N X(t+\tau) = J_N X(t) + J_N \triangle r(t) = J_N X(0) + \sum_{p=0}^{t/\tau} \triangle r(p\tau) = J_N r(t)$. Thus,

$$\begin{aligned}\delta(t+\tau) &= (I - h\mathcal{L}_{\mathcal{G}})\,\delta(t) + h\mathcal{L}_{\mathcal{G}} e(t) + \triangle r(t) - J_N \triangle r(t), \\ e(t+\tau) &= (I + h\mathcal{L}_{\mathcal{G}})\, e(t) - h\mathcal{L}_{\mathcal{G}}\delta(t) + \triangle r(t) - \epsilon Q\left[\epsilon^{-1}((I+h\mathcal{L}_{\mathcal{G}})e(t) - h\mathcal{L}_{\mathcal{G}}\delta(t) + \triangle r(t))\right],\end{aligned} \tag{B.1}$$

where $\delta(t), e(t)$ are defined in (5) and (17), respectively. Because $\mathcal{G}$ is balanced, by [5, Theorem 7] and Assumption (A4), we have $\lambda_2(\mathcal{L}_{\hat{\mathcal{G}}}) > 0$ and hence

$$\begin{aligned}\|\delta(t+\tau)\|^2 &= \delta^T(t)\left(I - 2h\mathcal{L}_{\hat{\mathcal{G}}} + h^2\mathcal{L}_{\mathcal{G}}^T\mathcal{L}_{\mathcal{G}}\right)\delta(t) + \|h\mathcal{L}_{\mathcal{G}} e(t) + \triangle r(t) - J_N \triangle r(t)\|^2 \\ &\quad + 2\delta^T(t)\left(I - h\mathcal{L}_{\mathcal{G}}^T\right)(h\mathcal{L}_{\mathcal{G}} e(t) + \triangle r(t) - J_N \triangle r(t)) \\ &\leqslant \left(1 - 2h\bar{\lambda}_0 + h^2\overline{L}^2 + \epsilon_3\right)\|\delta(t)\|^2 + \|h\mathcal{L}_{\mathcal{G}} e(t) + \triangle r(t) - J_N \triangle r(t)\|^2 \\ &\quad + \epsilon_3^{-1}\|\left(I - h\mathcal{L}_{\mathcal{G}}^T\right)(h\mathcal{L}_{\mathcal{G}} e(t) + \triangle r(t) - J_N \triangle r(t))\|^2 \\ &\leqslant \rho_{h,\epsilon_3}\|\delta(t)\|^2 + \left[\epsilon_3^{-1}\left(1 - 2h\bar{\lambda}_0 + h^2\overline{L}^2\right) + 1\right]\left(h\overline{L}\|e(t)\| + \sqrt{N}\tau\theta\right)^2,\end{aligned} \tag{B.2}$$

where ϵ_3 is a positive constant. From (B.1) and $K \geqslant K_2(h,\tau,\epsilon,\epsilon_3)$, it is obvious that no quantizer is saturated at the initial time. Assume that no quantizer is saturated at time $k = 0, \tau, \cdots, t$. Then, we will show that under the quantization level $K \geqslant K_2(h,\tau,\epsilon,\epsilon_3)$, no quantizer will be saturated at $t+\tau$. In the conduction later, we will use the upper bound estimate for $e(k)$, that is, $\|e(k)\|_\infty \leqslant \frac{1}{2}\epsilon, k = \tau, \cdots, t+\tau$. By (B.2) and Assumption (A3), we have

$$\begin{aligned}\|\delta(t+\tau)\|^2 &\leqslant (\rho_{h,\epsilon_3})^{t/\tau+1}\|\delta(0)\|^2 + (\rho_{h,\epsilon_3})^{t/\tau+1}\epsilon_3^{-1}\left(h\overline{L}\|e(0)\| + \sqrt{N}\tau\theta\right)^2 \\ &\quad + \sum_{k=0}^{t/\tau-1}(\rho_{h,\epsilon_3})^{k+1}\epsilon_3^{-1}\left(h\overline{L}\|e(t-k\tau)\| + \sqrt{N}\tau\theta\right)^2 \\ &\leqslant (\rho_{h,\epsilon_3})^{t/\tau+1}N\big[C_\delta^2 + \epsilon_3^{-1}(hC_x\overline{L} + \tau\theta)^2\big] + \frac{N\rho_{h,\epsilon_3}\epsilon_3^{-1}(h\epsilon\overline{L}/2 + \tau\theta)^2}{1-\rho_{h,\epsilon_3}}(1 - (\rho_{h,\epsilon_3})^{t/\tau}).\end{aligned} \tag{B.3}$$

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From this, it follows that

$$\begin{aligned}
&\epsilon^{-1}\left\|(I+h\mathcal{L}_{\mathcal{G}})e(t+\tau)-h\mathcal{L}_{\mathcal{G}}\delta(t+\tau)+\triangle r(t+\tau)\right\|_\infty\\
&\leqslant \epsilon^{-1}\|I+h\mathcal{L}_{\mathcal{G}}\|_\infty\|e(t+\tau)\|_\infty+\epsilon^{-1}h\|\mathcal{L}_{\mathcal{G}}\|\cdot\|\delta(t+\tau)\|+\epsilon^{-1}\|\triangle r(t+\tau)\|_\infty\\
&\leqslant \Theta_2+\frac{1}{2}<\lfloor\Theta_2\rfloor+\frac{3}{2}\leqslant K+\frac{1}{2},
\end{aligned}$$

where Θ_2 is defined in (18). This together with $K \geqslant K_2(h,\tau,\epsilon,\epsilon_3)$ implies that no quantizer is saturated at time $t+\tau$. From (10) and (B.3), we conclude that the dynamic consensus will have a steady error as $t\to\infty$, upper bounded by

$$\vartheta(h,\tau,\epsilon,\epsilon_3)=\max_{i\in\mathcal{V}}\limsup_{t\to\infty}\left|x_i(t)-\frac{1}{N}\sum_{j=1}^{N}x_j(t)\right|\leqslant\sqrt{N}(h\epsilon\overline{L}/2+\tau\theta)\left[\frac{\rho_{h,\epsilon_3}}{\epsilon_3(1-\rho_{h,\epsilon_3})}\right]^{\frac{1}{2}}.$$

Thus, by (19), we have (20). □

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